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Prepared by
Dr. Jean I. F. King, Project Manager
Geophysics Corporation of America
Bedford, Massachusetts

for

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National Aeronautics and Space Administration
Greenbelt, Maryland

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I. INTRODUCTION

The central problem of the Contract has been the formulation of an optimum inversion technique for inferring vertical temperature structure from remote radiometric observations. In order to focus on the difficulties inherent in the inversion process, let us assume the atmosphere is representable by a plane-parallel model in local thermodynamic equilibrium. The radiation intensity intercepted on vertical monochromatic viewing is given by the transfer equation as

$$I(\kappa) = \int_0^{\infty} B(u) e^{-\kappa u} \kappa du, \quad (1)$$

with B the Planck intensity, κ the monochromatic absorption coefficient, and u the mass cross-section of the absorber. Thus the atmosphere is a filter which transforms the temperature dependence with depth $B(u)$ into the variation of upwelling intensity with frequency $I(\kappa)$. We can write the formal solution for the temperature at once as the inverse Laplace transform of the intensity

$$B(u) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{I(\kappa)}{\kappa} e^{\kappa u} d\kappa. \quad (2)$$

The difficulty is that the intensity is not known as a continuous function, but is sensed only at certain frequencies. Since I is the transform of a physically meaningful temperature profile and therefore a smooth curve, the problem becomes one of constructing the optimum interpolation formula yielding the intensity $I(\kappa)$ from the partial know-

ledge $I = I_1$ at $\kappa = \kappa_1$.

Various functions have been tried. Perhaps the simplest is to fit the intensity curve with a sum of the inverse powers of the absorption coefficient κ

$$I(\kappa) = \kappa b(\kappa) \approx \sum_{j=0}^m \frac{a_j}{\kappa^j}, \quad (3)$$

where the superscript denotes the j th power of κ .

This is equivalent to approximating the source function by a power series as we see at once on performing the inversion

$$B(u) \approx \sum_{j=0}^m \frac{a_j u^j}{j!}. \quad (4)$$

Since the intensity is presumed known at the $m + 1$ frequencies ν_1 , Eqn. (3) becomes the following linear simultaneous equation set whose solution, or inversion, is readily accomplished

$$I(\kappa_1) = \sum_{j=0}^m \frac{a_j}{\kappa_1^j}. \quad (5)$$

There are two faults of this method. First, there is no assurance that the observed intensity profile can be reasonably fit by a finite sum of inverse powers of κ . This is equivalent to approximating the temperature distribution with a truncated power series. Secondly, no rationale is provided for the choice of terms in the power series.

Another function class used to generate an interpolation formula for the intensity is the quotient polynomial

$$I(\kappa) = \kappa b(\kappa) \approx \text{const} \frac{\prod_{\alpha=1}^{m-1} (\kappa + \kappa_{\alpha})}{\prod_{j=1}^m (\kappa + \kappa_j)} = \sum_{j=1}^m \frac{a_j}{1 + \frac{\kappa}{\kappa_j}} \quad (6)$$

An inversion shows the equivalence with an expansion of the temperature in a sum of exponential functions

$$B(u) = \text{const} - \sum_{j=1}^m a_j \exp(-\kappa_j u) \quad (7)$$

Given $2m$ values of the intensity, an algorithm exists for determining the constants a_j and κ_j . Thus the exponential function approximation is free of the second drawback. Its flaw, however, is the inability of a finite exponential sum to simulate the normally "bumpy" temperature profile.

Although we have treated two special cases thus far, similar considerations apply to other orthogonal polynomial expansions of the temperature such as Fourier series, Legendre, Laguerre and Chebyshev polynomials. As a matter of fact, the intensity profile for all these expansions including power series is expressible as a quotient polynomial with suitable specification of the κ_j .

Still missing is a rationale for choice among the function classes and, given that, a justification for choosing which terms in the sub-class

provide the best fit. In what follows we have devised a variable slab inversion technique which is free from these drawbacks and which represents, in some sense, the optimum inversion technique. The next section, which describes the method, appeared as a Note in the May 1964 issue of the Journal of the Atmospheric Sciences. Since the exposition was of necessity compressed, we shall develop the arguments leading to the paper at greater length.

Returning to Eqn. (1), we shall find it more convenient to treat the integral remaining after an integration by parts

$$I(\kappa) - B(0) = \int_0^{\infty} e^{-\kappa u} \frac{dB}{du} du \quad . \quad (8)$$

Using the fundamental theorem of integral calculus we can express the integral approximately as the sum

$$I(\kappa) - B(0) \approx \sum_{j=1}^m e^{-\kappa u_j} \Delta B_j \quad . \quad (9)$$

Thus in contrast to the previous methods which approximated the integrand $B(u)$, we approximate instead the integral by constructing a quadrature formula. The chief advantage is greater tolerance in the functional form of $B(u)$. The fundamental theorem assures convergence for large m and remains valid even for mild temperature discontinuities.

Eqn. (9) holds exactly for a temperature distribution consisting of isothermal slabs, i.e. sums of step functions. Since the derivative of

a unit step function is a delta function, the temperature gradient is a sum of delta functions. Thus we have

$$\frac{dB}{du} = \sum_{j=1}^m \Delta B_j \delta(u-u_j) \quad , \quad (10)$$

leading upon substitution into Eqn. (8) to

$$\begin{aligned} I(\kappa) - B(0) &= \sum_{j=1}^m \Delta B_j \int_0^{\infty} e^{-\kappa u} \delta(u-u_j) du \\ &= \sum_{j=1}^m e^{-\kappa u_j} \Delta B_j \quad , \end{aligned} \quad (11)$$

an exact result.

Having thus chosen the functional form of our interpolation formula for the intensity, we want now to invert the equation for the temperature profile. The first constraint is the requirement that the interpolation formula conform to the observed intensities

$$I(\kappa_1) - B(0) = \sum_{j=1}^m \Delta B_j \exp(-\kappa_1 u_j) \quad (12)$$

This is a simultaneous equations set in the $2m$ unknowns u_j and ΔB_j , with the u_j occurring nonlinearly as the argument of the exponential function. Clearly $2m$ intensity values $I(\kappa_1)$ are needed to determine the solutions.

One procedure for the solution converts (12) into a linear equation set by choosing m fixed values of u_j , usually in some evenly spaced fashion. In this manner the equations are transformed into the linear matrix form similar to Eqn. (5) since both κ_1 and u_j are now given

$$I(\kappa_1) - B(0) = \sum_{j=1}^m \Delta B_j \exp(-\kappa_1 u_j), \quad j=1, \dots, m. \quad (13)$$

The inadmissible feature of this procedure is that our a priori choice of slab boundaries u_j forces the temperature profile to conform to an arbitrarily imposed structure. Put otherwise, an isothermal slab has two degrees of freedom: its height given by ΔB_j and thickness by u_j . By predetermining the u_j 's we have robbed the temperature profile of this "thickness" degree of freedom. It would appear that this is too great a price to pay for a linearization of the problem.

It is natural to inquire if the nonlinear set (12) can be solved directly for the ΔB_j and u_j without imposing, in effect, the artificial constraint of predetermined thicknesses. The answer, after a simple transformation, is affirmative. With the substitution

$$x_j = \exp(-\kappa_0 u_j), \quad u_j = -\frac{\ln x_j}{\kappa_0}$$

the equation set (12) becomes

$$I(\kappa_1) - B(0) = \sum_{j=1}^m \Delta B_j x_j^{\kappa_1/\kappa_0} \quad (15)$$

Those familiar with numerical analysis will recognize the similarity to an equation set used in the construction of Gaussian quadrature formulas given, for example, in Chandrasekhar (p. S9)

$$\alpha_i = \sum_{j=1}^m a_j x_j^i, \quad i = 0, 1, \dots, 2m - 1 \quad (16)$$

The solution to this nonlinear set is given uniquely by an elegant algorithm in which the x_j 's, corresponding to the slab boundaries u_j are found as the roots of an algebraic equation of degree m . The weights a_j (our ΔB_j) then are readily found as the solution of a set of m linear equations. It is interesting to note that the uniqueness condition

$$i = \frac{\kappa_i}{\kappa_0} = 0, 1, \dots, 2m - 1 \quad (17)$$

specifies that we view the atmosphere at frequencies such that the absorption coefficients κ_i are integral multiples of some base value κ_0 . Thus our method provides a rationale for the choice of viewing frequencies for stability in the inversion process. Other techniques lack this specification, although one might be intuitively led to make the equal-interval choice here dictated.

Some element of choice remains in the base value taken for κ_0 . Ordinarily if one has $2m$ viewing channels, κ_0 would be taken as

$$\kappa_0 = \frac{\kappa_{\max}}{2m-1}, \quad (18)$$

$$\text{with } i = \frac{K_1}{K_0} = 0, 1, \dots, 2m - 1, \quad (19)$$

as required by the formula.

In conclusion we have devised an inversion technique which yields the unique configuration of m isothermal slabs of heights B_j and thicknesses Δu_j which conform to $2m$ intensity observations. The main advantage of the technique is that the structure of the inferred temperature profile is determined solely by the data, and not by any a priori choice of fitting polynomials.

The close connection between this inversion technique and the Gaussian quadrature formula is not fortuitous. It is generic. Gauss showed that a strategic choice of m points within an integration interval leads to a more accurate numerical quadrature as compared to the equally-spaced intervals of the Newton-Cotes method. In our inversion problem which is akin to numerical differentiation, the need for variable slab thicknesses for optimum fitting is even more compelling.

II. RESEARCH DURING REPORT PERIOD

In the inversion problem an algorithm is sought for the inference of atmospheric vertical thermal structure from remotely-sensed radiometric observations. To accomplish this we propose a new inversion method, the variable slab technique, and demonstrate its application by two illustrative examples.

All inversion procedures attempt to recover the thermal profile from observations of the upwelling intensity $I(\kappa/\mu)$ at various directions and/or frequencies. Transfer theory specifies the temperature dependence on depth as the solution of a linear integral equation

$$I(\kappa/\mu) = - \int_0^{\infty} B(u) \frac{\partial \mathfrak{S}}{\partial u} du, \quad (20)$$

where B is the Planck intensity considered here an implicit function of absorber depth u , and \mathfrak{S} is the kernel transmittance averaged over a narrow frequency interval

$$\mathfrak{S}(\kappa u/\mu) = \frac{1}{\Delta\nu} \int_{\Delta\nu} e^{-\kappa_\nu u/\mu} d\nu, \quad (21)$$

with κ_ν the monochromatic absorption coefficient.

For our purposes a simplified gray, plane-parallel, fixed-frequency model suffices in which the intensity is scanned over nadir angle $\theta = \cos^{-1}\mu$. Under these conditions the intensity is a Laplace transform of the indicial function

$$I(1/\mu) = \int_0^{\infty} B(\tau) e^{-\tau/\mu} d\tau/\mu = \frac{b(1/\mu)}{\mu}, \quad (22)$$

where we have transformed to the new variable, optical depth $\tau = \kappa u$.

Conventionally the temperature profile is approximated by an appropriate series expansion

$$B(\tau) = \sum_j a_j F_j(\tau) \quad . \quad (23)$$

The series need not be orthogonal but should converge to the exact solution.

The substitution of this expansion into Equation (22) identifies the intensity with the Laplace transforms of the indicial approximation

$$\mu I(1/\mu) = \sum_j a_j f_j(1/\mu) \quad . \quad (24)$$

Intensity observations at n discrete directions enable one to determine n coefficients of the temperature expansion, Equation (23), as the solution of the linear simultaneous equation set

$$\mu_i I(1/\mu_i) = \sum_j f_{ij} a_j \quad , \quad i = 1, 2, \dots, n \quad . \quad (25)$$

A variety of different function classes have been used in inversion attempts. Examples are power series (King 1959), exponential functions (King 1964), and various orthogonal sets such as Legendre, Chebyshev, or Laguerre polynomials (Yamamoto 1961). All these expansions share a common defect rendering them unsuitable for the inversion procedure. By choosing a particular finite polynomial expansion we restrict the form of the thermal profile.

Standing in contrast to this analytic procedure in which the temperature is broken down into components, is the synthetic method which approximates the profile by isothermal slabs. This is expressed by expanding the lapse-rate in a sum of delta functions

$$\frac{dB(\tau)}{d\tau} = \sum_j (\Delta B)_j \delta(\tau - \tau_j) \quad . \quad (26)$$

Proceeding as before, the substitution of this slab approximation into the transfer equation yields the equation set

$$\begin{aligned} I(1/\mu) - B(0) &= \int_0^\infty \frac{dB(\tau)}{d\tau} e^{-\tau/\mu} d\tau \\ &= \sum_j (\Delta B)_j e^{-\tau_j/\mu} \quad . \end{aligned} \quad (27)$$

We have not specified the slab boundaries τ_j . Heretofore these positions have been assigned in advance, usually at significant levels in the atmosphere where lapse-rate discontinuities are anticipated (Kaplan 1959, Wark 1961). Once again a knowledge of the intensity profile at the n directions μ_i leads to a linear equation set determining the slab temperatures at n preset intervals

$$I(1/\mu_i) - B(0) = \sum_j (\Delta B)_j \exp(-\tau_j/\mu_i) \quad .$$

As we shall see this synthetic method is extremely sensitive to the choice of slab boundaries. As with the analytic procedure, the same criticism holds. The choice of τ_j is critical, forcing in advance a particular structure on the slab profile.

We propose, therefore, a variable or floating slab method which determines uniquely the slab strengths and thicknesses for a given intensity profile. Consider Equation (28). With the substitution $x_j = \exp(-\tau_j)$ we succeed to a set of nonlinear simultaneous equations each of degree $1/\mu_i$

$$I(1/\mu_i) - B(0) = \sum_j (\Delta B)_j x_j^{1/\mu_i} . \quad (29)$$

By choosing the sequence of viewing directions

$$\frac{1}{\mu_i} = i = 0, 1, 2, \dots, 2n-1 \quad (30)$$

we obtain the equation set of successively higher degree

$$\alpha_i = \sum_j a_j x_j^i , \quad i = 0, 1, 2, \dots, 2n-1 \quad (31)$$

where we have written $\alpha_i = I(i) - B(0)$ and a_j for $(\Delta B)_j$.

The equation set arises in the construction of quadrature formulas of the Gaussian type (Lanczos 1956, Kopal 1961). Despite its nonlinearity the set is soluble uniquely by an elegant algorithm given, for example, by Chandrasekhar (1950) which consists of three steps. First, n auxiliary constants d_i are determined from the linear equation set

$$\alpha_{i+n} + \sum_{\ell=0}^{n-1} c_\ell \alpha_{i+\ell} = 0 , \quad i = 0, 1, \dots, n-1 . \quad (32)$$

The slab boundaries $\tau_j = -\ln x_j$ are then obtained as the n roots of the equation

$$x^n + \sum_{\ell=0}^{n-1} c_{\ell} x^{\ell} = 0 \quad . \quad (33)$$

The knowledge of the roots enables one to determine the weights a_j from the first n equations of the set (31).

The solution admits the interpretation: Given n isothermal slabs, the b_j and $\tau_j = -\ln x_j$ are the unique choice of slab weights and thicknesses fitting the $2n$ intensity observations.

Figure 1 displays three and five slab atmospheres ingerrred from intensity values of the following model atmosphere

$$I(1/\mu_i) = \frac{\mu_i}{\mu_i + 1} \quad , \quad \frac{1}{\mu_i} = \begin{cases} 0, 1, \dots, 5 \\ 0, 1, \dots, 9 \end{cases} \quad (34)$$

The solid curve is the exact solution obtained directly by inversion

$$\begin{aligned} B(\tau) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mu I(1/\mu) e^{\tau/\mu} d(1/\mu) \\ &= 1 - e^{-\tau} \quad . \end{aligned} \quad (35)$$

The slab approximation is impressive. Note that the slab thickness is smallest in the region of greatest slope, thus minimizing the "cornering" error.

For comparison the same ten intensity values were used to infer the ten weights of slabs bounded at the ten preset intervals

$$\tau_j = 0.1, 0.2, \dots, 1.0 \quad . \quad (36)$$

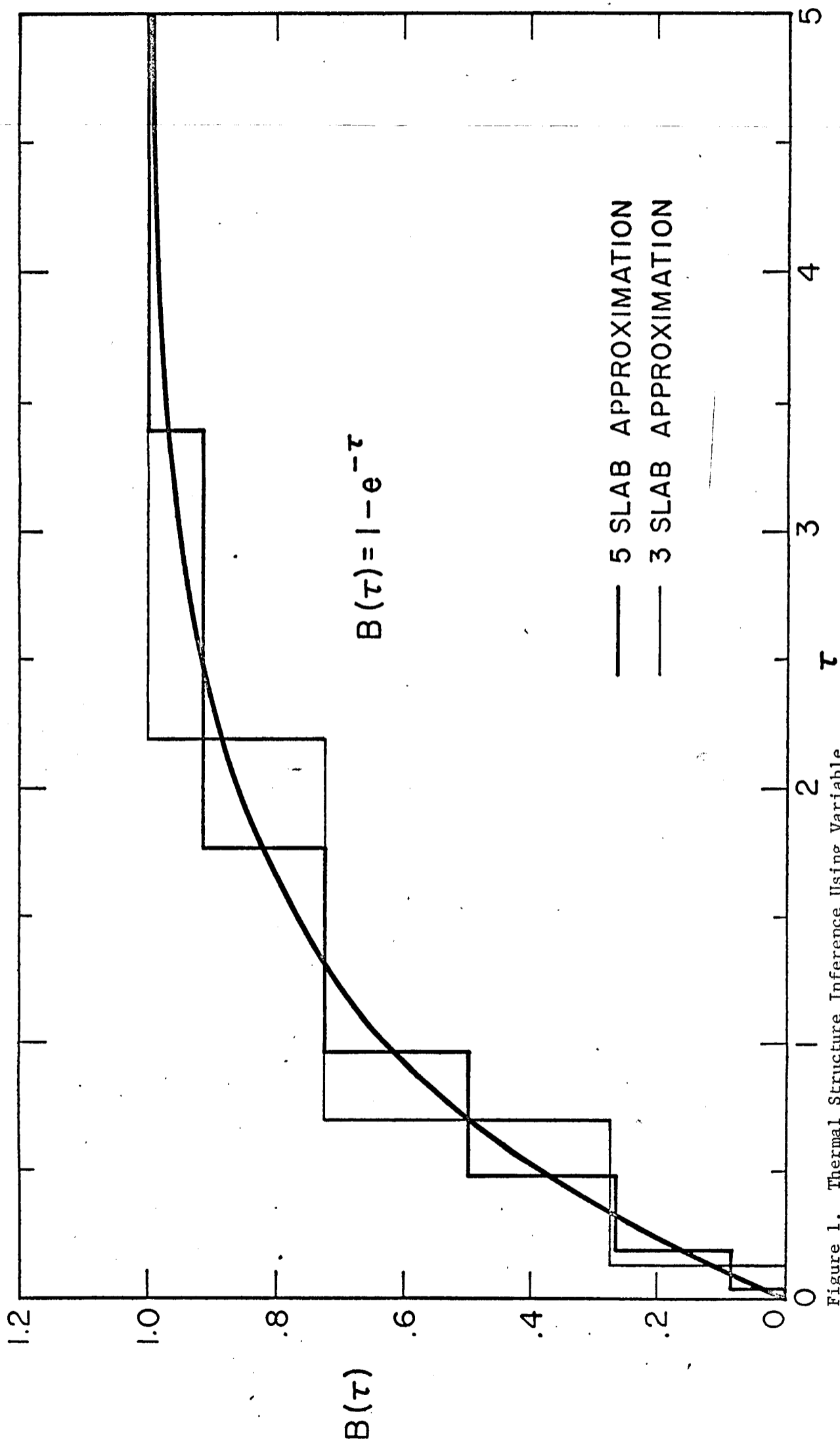


Figure 1. Thermal Structure Inference Using Variable Slab Method.

The thermal structure inferred by solving the equation set (28) for this model is grossly unrealistic (see Table 1).

The superiority of the floating over the fixed slab method can be understood by its relation to quadrature formulas. The Gaussian quadrature method achieves more accuracy than the present Newton-Cotes intervals by allowing the integrand thicknesses to vary. Similarly in our inverse problem we have the additional degree of freedom in the determination of the slab boundaries as the unique solution of Eqn. (33).

A second, more complicated atmospheric thermal profile is inferred in Figure 2 using the ten intensity values

$$I(1/\mu_i) = \frac{\mu_i}{(\mu_i + 2)^2}, \quad \frac{1}{\mu_i} = 1, 2, \dots, 10 \quad (37)$$

The agreement of the inferred slab model to the exact solution

$$B(\tau) = \tau e^{-2\tau} \quad (38)$$

is remarkable for optical depths less than $\tau \approx 1$. For large values of optical depth the divergence is expected since even the deepest sensing ($\mu_i = 1$) gives little information on the atmosphere beyond unit optical depth.

The constant slab slope ($dB/d\tau = -0.14$) in Figure 2 arises from the stipulation that the weights $(\Delta B)_j$ in the Gaussian quadrature formula, Equation (29), be positive (Kopal, 1961). This requirement can always be satisfied by adding a constant slope to the lapse-rate which is sub-

TABLE 1

Thermal Profile for Constant Thickness Slabs

τ	$B(\tau)$
0 - 0.1	0
0.1 - 0.2	-82
0.2 - 0.3	+672
0.3 - 0.4	-1314
0.4 - 0.5	-1331
0.5 - 0.6	+3167
0.6 - 0.7	+2853
0.7 - 0.8	-4815
0.8 - 0.9	-1250
0.9 - 1.0	+2109
> 1.0	0

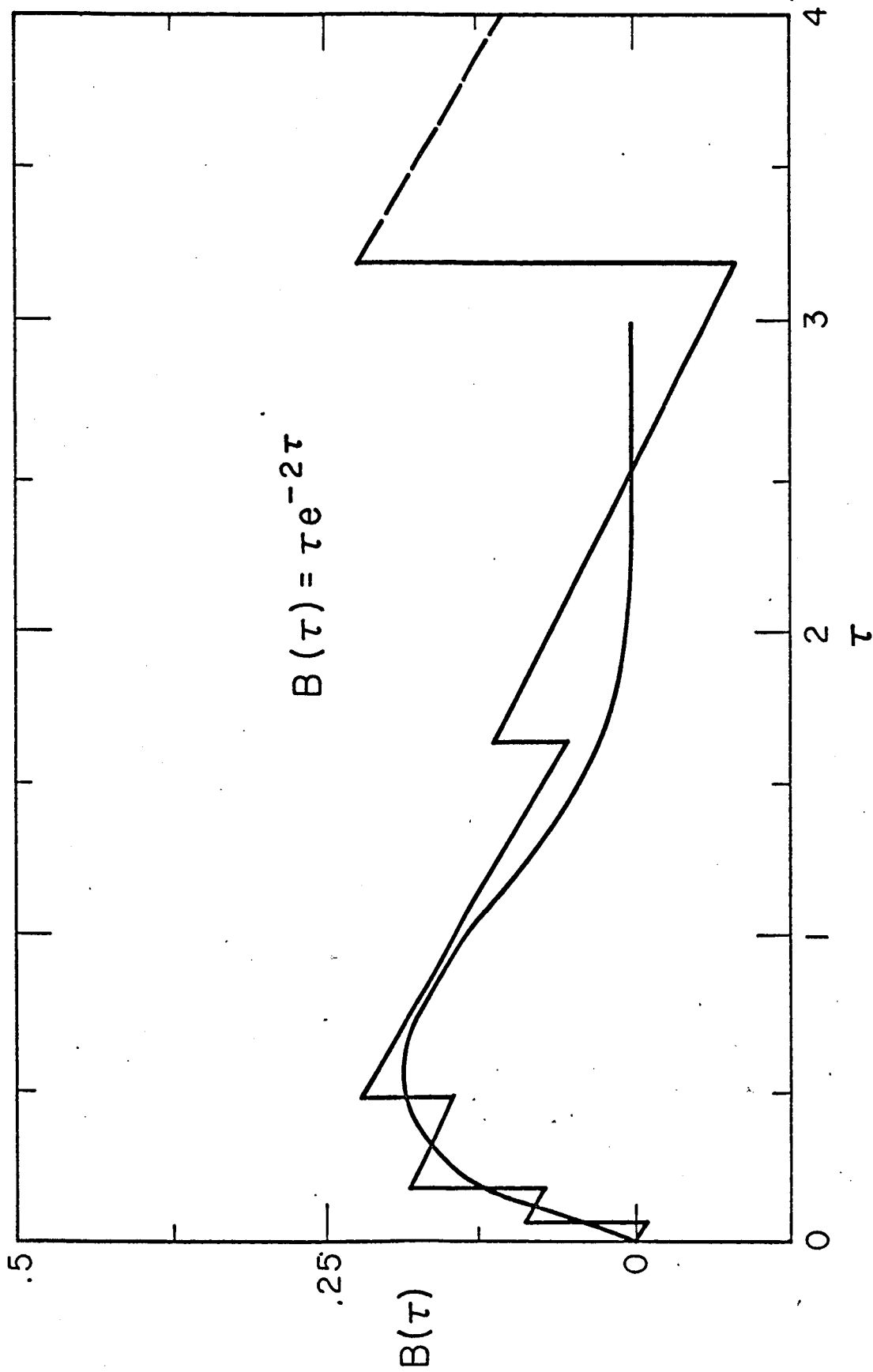


Figure 2. Thermal Structure Inference Using Variable Slab Method.

sequently subtracted after the inversion operation.

In a forthcoming paper "Meteorological Inferences from Satellite Radiometry, II" this floating slab method will be applied to thermal inferences of synthetic atmospheric models. An extension of the formalism to treat arbitrary band transmittance kernels is planned. The implications of the technique for the error analysis of raw radiometric data will be discussed.

III. FORTHCOMING QUARTERLY RESEARCH PROGRAM

The hallmark of a good theory is the light shed on problems beyond those which brought it into being. An example is this Gaussian inversion method which, almost incidentally, provides a rationale for the choice of viewing frequencies heretofore lacking. In fairness it should also be noted that a successful theory in replying to a problem, poses new questions as well. Three of these are:

- 1) The uniqueness problem. The elegant algorithm alluded to earlier works only if sequential integral values for κ_1/κ_0 . What if one of a sequence is missing? What happens if non-integral values of κ_1/κ_0 are taken? Do multivalued solutions result? Does the numerical inversion become unstable?
- 2) Generalization to constant lapse-rate slabs. Thus far our interpolation formula for $I(\kappa)$ has been synthesized from isothermal slabs of varying thicknesses. Is it possible to synthesize a profile from constant lapse-rate slabs? What additional requirements in accuracy does this entail?
- 3) Generalization to non-gray atmospheres. The formalism to date has dealt only with transmittances of an exponential function form. How is this altered if more realistic models are used?

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